11/5/21

Calculus III

16.1: Vector Fields

One Last Spherical Example

Ex: Compute the value of ^{a} closed disk of radius $\alpha > 0$.

NB: We did this already in cartesian coordinates, but it was nasty...

Sol (Spherical Coordinates): $D\alpha = \{(\rho, \theta, \varphi) : 0 \leq \rho \leq \alpha,$

$0 \leq \theta \leq 2\pi,\ 0 \leq \varphi \leq \pi\}$

$$\text{Vol}(D\alpha) = \iiint_{\substack{D\alpha \\ \text{cart}}} 1\, dV$$

$$= \iiint_{\substack{D\alpha \\ \text{spherical}}} 1 \cdot \rho^2 \sin(\varphi)\, dV_{\text{sph.}}$$

$$= \int_{\rho=0}^{\alpha} \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi} \rho^2 \sin(\varphi)\, d\varphi\, d\theta\, d\rho$$

$$= \int_{\rho=0}^{\alpha} \int_{\theta=0}^{2\pi} -\rho^2 \Big[\cos(\varphi)\Big]_{\varphi=0}^{\pi} d\theta\, d\rho$$

$$= \int_{\rho=0}^{\alpha} \int_{\theta=0}^{2\pi} -\rho^2(-1-1)\, d\theta\, d\rho$$

$$= 2\int_{\rho=0}^{\alpha} \int_{\theta=0}^{2\pi} \rho^2\, d\theta\, d\rho$$

$$= 2\int_{\rho=0}^{\alpha} \rho^2 \left[\theta\right]_{\theta=0}^{2\pi} \, d\rho$$

$$= 4\pi \int_{\rho=0}^{\alpha} \rho^2 \, d\rho$$

$$= \frac{4}{3}\pi \left[\rho^3\right]_{\rho=0}^{\alpha}$$

$$= \frac{4}{3}\pi (\alpha^3 - 0)$$

$$= \frac{4}{3}\pi \alpha^3$$

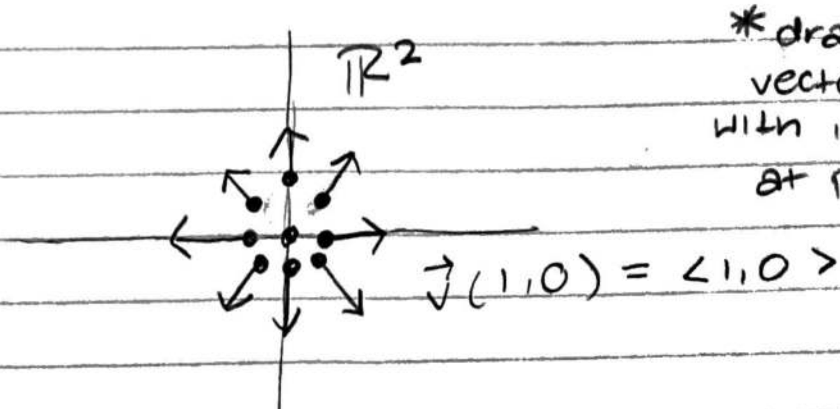
Vector Fields

Goal: Study functions $\vec{V} : \mathbb{R}^n \to \mathbb{R}^n$
(most of the time $n = 2$ or $n = 3$ for us)

Defn: A vector field in $\mathbb{R}^n$ is a function
$\vec{V} : \mathbb{R}^n \to \mathbb{R}^n$

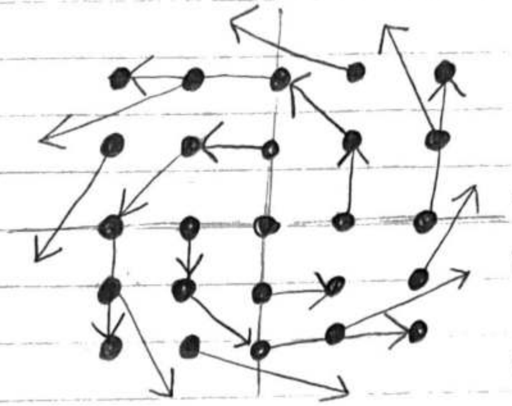
Ex: Consider vector field $\vec{V}(x,y) = \langle x, y \rangle$ on $\mathbb{R}^2$. How do we visualize it?

↳ Every point in $\mathbb{R}^2$ has an associated vector attached to it by this vector field:



* draw the vector $\vec{V}(x,y)$ with its tail at point (x,y) *

Ex: Draw $\vec{v}(x,y) = \langle -y, x \rangle$.



$v(1,0) = \langle 0,1 \rangle$
$v(2,0) = \langle 0,2 \rangle$
$v(-1,0) = \langle 0,-1 \rangle$
$v(-2,0) = \langle 0,-2 \rangle$
$v(-1,1) = \langle -1,-1 \rangle$
$v(0,1) = \langle -1,0 \rangle$
$v(-2,1) = \langle -1,-2 \rangle$

Ex: The gradient of a function is always a vector field.
↳ e.g. for $f(x,y) = xy$, $\nabla f = \langle y, x \rangle$ is a vector field on $\mathbb{R}^2$

vector field = "v.f."

- A vector field like this is sometimes called a "gradient vector field".
  e.g. $f(x,y,z) = e^{x+y^2}\cos(z+x)$
  $\nabla f = \langle e^{x+y^2}\cos(z+x) - e^{x+y^2}\sin(z+x),\ 2y\cos(z+x)e^{x+y^2},\ -e^{x+y^2}\sin(z+x) \rangle$
  is a vector field $\vec{\nabla f}(x,y,z)$

• Obvious Question: How do we know when a vector field is a gradient vector field?
↳ is every v.f. a grad. v.f.?

Terminology: A conservative vector field is a gradient v.f.

If $\vec{V} = \nabla f$ is a conservative v.f., we call f a potential function for $\vec{V}$

Now, is every v.f. conservative?

In $\mathbb{R}^2$, a conservative v.f. has form $\vec{v} = \langle f_x(x,y), f_y(x,y) \rangle$ for some potential function f. By Clairaut's Theorem, $f_{xy} = f_{yx}$, so every conservative v.f. $\vec{v} = \langle \alpha(x,y), \beta(x,y) \rangle$ has to satisfy $\alpha_y = \beta_x$.

Ex: $\vec{v} = \langle \overset{v_x}{-y}, \overset{v_y}{x} \rangle$ has

$$\frac{d}{dy}[v_x] = \frac{d}{dy}[-y] = -1$$

On the other hand...

$$\frac{d}{dx}[v_y] = \frac{d}{dx}[x] = 1$$

$\therefore \vec{v}$ is non conservative lest it violates Clairaut's Thereom

$\therefore$ Not every vector field is a gradient vector field. !!

Prop: A vector field $\vec{v} = \langle v_{x_1}, v_{x_2}, \dots, v_{x_n} \rangle$ is conservative if and only if $\frac{d}{dx_i}[v_{x_j}] = \frac{d}{dx_j}[v_{x_i}]$

for all i,j. (i.e. a v.f. is conservative if and only if it satisfies Clairaut's Thereom)

Ex: Is $\vec{v} = \langle x, y \rangle$ conservative?

Sol: $$\frac{dv_x}{dy} = \frac{d}{dy}[x] = 0$$

$$\frac{dv_y}{dx} = \frac{d}{dx}[y] = 0$$

$\therefore$ by the proposition $\vec{V} = \langle x, y \rangle$ is conservative:
What is it's potential function?
By conservativity, $\nabla f = \vec{V}$ for some function $f(x,y)$. i.e. $f_x(x,y) = x$, and $f_y(x,y) = y$.

B/C $\frac{df}{dx} = x$, we know $f(x,y) = \int \frac{df}{dx} dx = \int x dx$

$= \frac{1}{2}x^2 + c(y)$

$\therefore y = \frac{df}{dy} = \frac{d}{dy}\left[\frac{1}{2}x^2 + c(y)\right] = \frac{dc}{dy}$.

$\therefore c(y) = \int \frac{dc}{dy} dy = \int y dy = \frac{1}{2}y^2 + D$ ← An actual constant (honest constant)

Hence $f(x,y) = \frac{1}{2}x^2 + c(y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + D$ ← for any constant

D is a potential function of $\vec{V}$

Ex: $\vec{V} = \langle 2xy, x^2 - 3y^2 \rangle$. Conservative? If yes, potential?

Sol: $\frac{dV_y}{dx} = \frac{d}{dx}[x^2 - 3y^2] = 2x$

$\frac{dV_x}{dy} = \frac{d}{dy}[2xy] = 2x$

$\therefore \vec{V}$ is conservative

If $\vec{V} = \nabla f$, the $f_x = 2xy$

$f = \int f_x dx = \int 2xy\, dx = x^2y + c(y)$
$\therefore f_y = \frac{d}{dy}[x^2y + c(y)]$

$\cancel{x^2} - 3y^2 = \cancel{x^2} + c'(y)$

$\therefore c'(y) = -3y^2$

$\therefore c(y) = \int -3y^2\, dy = -y^3 + D$

$\therefore f(x,y) = x^2 y + c(y) = x^2 y - y^3 + D$ ← is a potential for all constant D

NB: The method for computing the potentials can be used to prove the proposition from earlier...

Ex: $\vec{v} = \langle \ln(x+y), e^{x+y} + \frac{1}{x+y} \rangle$

Conservative? If yes, potential?

Sol: $\frac{dV_x}{dy} = \frac{d}{dy}[\ln(x+y)] = \frac{1}{x+y}$

$\frac{dV_y}{dx} = \frac{d}{dx}\left[e^{x+y} + \frac{1}{x+y}\right] = e^{x+y} - (x+y)^{-2}$

$\therefore \vec{v}$ is not conservative, $\therefore$ no potential.

Last Time: You saw

$$\int_{y=c}^{d} \int_{x=a}^{b} f(x) g(y)\, dx\, dy = \int_{x=a}^{b} f(x)\, dx \cdot \int_{y=c}^{d} g(y)\, dy$$

↑ only true when:

① Integrating over a rectangle $[a,b] \times [c,d]$

② Integrated as a "separable function" i.e. $h(x,y) = f(x) \cdot g(y)$